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Designing a stabilizing controller for discrete-time nonlinear feedforward systems with unknown input saturation

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Abstract

This article aims to design a low gain feedback controller for a class of discrete-time feedforward nonlinear systems (DFNSs) with unknown input saturation. Although the design of the low gain feedback control can be converted into the choosing of a parameter, it is difficult to design the control due to the inherent nonlinear dynamics and the unknown input constants. To solve this problem, we first show that the low gain feedback control can be found to stabilize the DFNSs when the saturation is known. Then, by analyzing the dynamics of DFNSs under the saturating input, the relation is built between the parameter and the system state. Finally, we introduce a technology to update the parameter based on different performance, whose effectiveness is illustrated through considering a numerical example.

KEYWORDS

discrete-time system, dynamic gain feedback control, feedforward system, low gain feedback control, nonlinear system

1 | INTRODUCTION

The feedforward nonlinear system represents a special class of nonlinear systems, and many results have been devoting to its control problems. This is due to some physical devices in feedforward dynamics, such as “cart-pendulum” system, the “ball and beam” with a friction term. Another key aspect is that, from a theoretical point of view, the feedforward systems are in general not feedback linearizable, which present some challenges to design stabilizing control. In this way, the last decades have been developing various technologies, such as the saturation-based method,[1,2] forwarding method,[3] and the low gain feedback control method[4-9] to stabilize the feedforward nonlinear systems. However, most of these works on this subject are concerned with continuous-time systems. The corresponding results on discrete-time feedforward nonlinear systems (DFNSs) have been really lagged.

Due to the Lyapunov equation is in different forms between the discrete-time case and the continuous-time case, the stabilization of DFNSs presents new challenges. The DFNSs can be derived by integrating the continuous-time equations with Brunovsky structure or feedforward structure. But compared with other results on discrete-time systems,[10,11] little results are focusing on the discrete-time system in feedforward forms. Early stabilization problem was focusing on the case when the nonlinearities were governed by a stabilizable condition.[12,13] A recent work was developing the saturation methodology to design the stabilizing controller when the system nonlinearities satisfied the quadratic growth condition.[14] It is noticed that the low gain feedback control method is one of the dominating technologies in the study of continuous-time feedforward nonlinear systems. The low gain feedback control method can design a proportional control, which is easy to be implemented in practical systems. Furthermore, due to its simple form, the parameter is easy to be turned, and the turning mechanism is easy to understand. The low gain feedback control was designed to stabilize feedforward nonlinear system,[4,6] followed by the key contributions developed into various forms for specific control problems.[5,7,15,16] It should be mentioned that the sampled-data controller was designed for the feedforward nonlinear systems based on the low gain feedback control.[17] And the discrete-time low gain feedback control was developed for linear systems.[18-20] But, for directly analyzing the DFNSs, the low gain feedback control has not been developed, which remains an open subject.

The saturating control is closely related to the low gain feedback control. For a continuous-time feedforward nonlinear system, the stabilizing control was designed through initiative introducing the saturating terms.[1,2] In some scenarios, the input saturation is required by the physical devices,[21] and the constraints may not be given precisely. The designed low gain feedback control may not pass the saturating restrictions, which degrades the effectiveness. An approximating function was introduced to estimate the saturating constraints, and then the control was designed based on the estimation.[21-23] Since the low gain feedback is not allowing larger changes,[15] how to design the low gain feedback control to dominate the unknown saturation?

In this article, we are going to design the low gain feedback control for the DFNSs whose nonlinear terms are governed by an input-depended growth condition. The growth rate can be the function of the input, which brings difficulties to the design. Meanwhile, the input saturation is considered, but the bounded is assumed as unknown. The key to this objective is to consider the three tasks: (i) Is there a low gain feedback controller to stabilize the DFNSs? (ii) How to update the control gains such that the control can pass the unknown saturation restrictions?, and (iii) How to evaluate the system performance and terminate the updating process? The contributions can be summarized as below:

- The DFNS will be stabilized by providing a low gain feedback control. The considered nonlinear terms have an input-depended growth rate, which is different from the existing system.[14] We employ the low gain feedback control technology to study the DFNSs, and the analysis is different from the continuous Lyapunov theory-based framework. Thus, exploring the low gain feedback control for DFNSs is different from the existing works.
- The low gain feedback control is studied with respect to the unknown saturation restrictions. Although the input saturation is closely related to the low gain feedback control, the unknown restrictions make the design more difficult. Through analyzing the system dynamics, we build the estimation for the system evolution, and give a new policy to update the control parameter.
- A new framework is provided to update the control parameter. The low gain feedback control was updated adaptively,[5] or improved as the time-varying control.[15] Different from the existing results, we introduce a new framework to update the parameter and evaluate the performance to determine whether the parameter is needed to be updated.

The remainder of this article is organized as follows. We will describe the problem in Section 2. The low gain feedback control will be designed in Section 3 when the saturation is not happen, and Section 4 will present an improved low gain feedback control to dominate the unknown saturation. The numerical example is going to illustrate the main results in Section 5. Some ending remarks will be summarized in Section 6, while a reference list will end this article.

Notations: We employ $\|\cdot\|$ to denote the Euclidean norm for vectors, or the induced Euclidean norm for matrices. The notation $\lceil x \rceil$ gives the greatest integer which is less than or equal to x. For matrix P, P^T represents its transpose, and $\lambda_{\max}(P)$, $\lambda_{\min}(P)$ denote the largest eigenvalue and the smallest eigenvalue of the matrix P, respectively. We use I to denote an $n \times n$ identity matrix.

2 | PROBLEM FORMULATION

2.1 | The system description

The studied DFNS is represented as

$$\begin{aligned} x_1(k+1) &= x_1(k) + p_1 x_2(k) + f_1(x(k), u(v(k))), \\ x_2(k+1) &= x_2(k) + p_2 x_3(k) + f_2(x(k), u(v(k))), \\ &\vdots \\ x_n(k+1) &= x_n(k) + p_n u(v(k)) + f_n(x(k), u(v(k))), \end{aligned} \tag{1}$$

where $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ is the system state, $v \in \mathbb{R}$ is the system input to be designed, and $u(v)$ denotes the input subject to saturation nonlinearity described by

$$u(v) = \mathrm{sat}(v) = \begin{cases} \mathrm{sign}(v)u_{\max}, & |v| \geq u_{\max}, \\ v, & |v| \leq u_{\max}, \end{cases}$$

where $u_{\max}$ is an unknown parameter of input saturation. Without loss of generality, constants $p_1, p_2, \dots, p_n$ are assumed to be positive. The initial index is assumed as 0, and the initial system state is $x(0) \in \Omega$ with Ω being a closed set in $\mathbb{R}^n$.

Remark 1. The constants p_1 to p_n can be any nonzero constant. When some of them are negative, we can make a state change to ensure they be positive. For example, consider p_i is the first negative number (p_1 to p_{i-1} is/are positive), then we define a new state $\tilde{x}_{i+1} = -x_{i+1}$ ($x_{n+1} = v$), and get a positive coefficient $-p_i$. Thus, we can assume all constants p_1 to p_n are positive.

The functions $f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)$ are continuous, and satisfy the following assumption.

Assumption 1. For any $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, $u \in \mathbb{R}$, it holds

$$|f_i(x, u)| \leq \phi(u)\left(|x_{i+2}| + |x_{i+3}| + \cdots + |u|\right), \quad i = 1, 2, \dots, n-2,$$

where $\phi(u)$ is an unknown continuous function with respect to u. Additionally, it holds $|f_{n-1}(x, u)| \leq \phi(u)\,|u|$ and $f_n(x, u) = 0$.

Remark 2. System (1) is in the feedforward form, which contains the linear part and the nonlinear part. If we do not consider the nonlinear part, that is, $f_i = 0$, it is the integrator system.[24] The nonlinear terms satisfying Assumption 1 has an input-depended growth rate, which was similarly considered for the continuous-time systems.[5] For the DFNSs, it was studied[14] when the nonlinear function f_i satisfies $|f_i| \leq M(x_{i+1}^2 + \cdots + x_n^2)$ in a given neighborhood $|x_j| \leq 1, j = i+1, \dots, n$ for a constant M. This is essentially different from Assumption 1. Thus, the control problem of system (1) has not been solved before.

2.2 | The low gain feedback control

To stabilize system (1), a candidate control can be expressed as

$$v(k) = -k_1 \frac{x_1(k)}{h^n} - k_2 \frac{x_2(k)}{h^{n-1}} - \cdots - k_n \frac{x_n(k)}{h}, \tag{2}$$

where k_1 to k_n are chosen later. Parameter $h \geq 1$ is a constant to be determined.

The controller (2) is parameterized by the parameter h. This control is called as the low gain feedback control, since it goes to zero as the parameter h increases. Although the low gain feedback control was initially proposed[25] to solve the semi-global stabilizing problem for systems subject to input saturation, it has not been employed to control the DFNSs with unknown saturation. To determine the parameter h, we employ the following lemma about the constants h_1 to h_n to analyze the system convergence.

Lemma 1 (5,6). *Let*

$$A = \begin{bmatrix} 0 & p_1 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \cdots & p_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_n \end{bmatrix}. \tag{3}$$

There exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$, a vector $K = [k_1, k_2, \dots, k_n] \in \mathbb{R}^n$, and a positive constant α such that

$$(A - BK)^T P + P(A - BK) \leq -\alpha P. \tag{4}$$

Remark 3. This lemma is based on the continuous Lyapunov equation (4). But, we will show that the control of the discrete-time system (1) can also be designed through this equation.

2.3 | The problem to be addressed

With the help of notation A, B in Lemma 1, we rewrite the closed-loop system (1) and (2) as

$$x(k+1) = (I + A)\,x(k) - Bu(K(h)x(k)) + F\,(k)\,, \tag{5}$$

where

$$K(h) = \left[k_1 \frac{1}{h^n}, k_2 \frac{1}{h^{n-1}}, \ \dots \ , k_n \frac{1}{h}\right], \quad F\,(k) = \begin{bmatrix} f_1\,(\cdot) \\ \vdots \\ f_n\,(\cdot) \end{bmatrix}.$$

Our problem in this article is to find the constant h such that the state of the closed-loop system (5) satisfies

$$\lim_{k \to +\infty} \|x(k)\| = 0.$$

To continue, since $\phi(u)$ is a continuous function on a closed set $[-u_{\max}, u_{\max}]$, we can re-express Assumption 1 as below.

Assumption 1*. For any $x = [x_1, x_2, \ \dots \ , x_n] \in \mathbb{R}^n$, $u \in \mathbb{R}$, there exists an unknown constant $\overline{\phi}$ such that

$$\begin{aligned} &|f_i(x,u)| \le \overline{\phi}\,(|x_{i+2}| + |x_{i+3}| + \cdots + |u|)\,, \quad i = 1, 2, \ \dots \ , n-2, \\ &|f_{n-1}(x,u)| \le \overline{\phi}\,|u|\,, \quad f_n(x,u) = 0, \end{aligned}$$

hold.

3 | SEMI-GLOBAL STABILIZATION WITH KNOWN SATURATION

In this section, we show that for any initial state $x(0) \in \Omega$, the parameter h can be found to get the stabilizing control (2). Consider the state transformation

$$z_1(k) = \frac{x_1(k)}{h^n}, \ z_2(k) = \frac{x_2(k)}{h^{n-1}}, \ \dots \ , \ z_n(k) = \frac{x_n(k)}{h}, \tag{6}$$

and we can get

$$\begin{aligned} z_1(k+1) &= z_1(k) + p_1 \frac{1}{h} z_2(k) + \frac{1}{h^n} f_1(x(k), u(k)), \\ &\ \vdots \\ z_n(k+1) &= z_n(k) + p_n \frac{1}{h} u(k) + \frac{1}{h} f_n(x(k), u(k)). \end{aligned}$$

Consider the control $v(k)$ can pass the saturation, and we can express the control $u(k)$ in (2) as

$$u(k) = -Kz(k), \tag{7}$$

where $z = [z_1, z_2, \ \dots \ , z_n]^T$. Then, following the notation in (3), the closed-loop system (5) can be rewritten as

$$z(k+1) = \left(I + \frac{1}{h} A - \frac{1}{h} BK\right) z(k) + \tilde{F}(k), \tag{8}$$

where $\tilde{F}(k) = \left[\frac{1}{h^n} f_1(x(k), u(k)), \ \dots \ , \frac{1}{h} f_n(x(k), u(k))\right]^T$.

Since h is a constant, the convergence of system (8) is equivalent to that of system (5). Thus, the following in this section is to analyze system (8), and we summarize the main result as below.

Theorem 1. *Let Assumption 1 holds and $u_{\max}$ being known. For any initial state $x(0) \in \Omega$, there exists a constant $h_{\min}$ such that if the parameter h in (2) satisfying $h \geq h_{\min}$, the state of closed-loop system (5) converges to 0, that is*

$$\lim_{k \to +\infty} \|x(k)\| = 0.$$

Proof. We first analyze the case when $v(t) \leq u_{\max}$. After the state transformation (6), the closed-loop system (1), (7) can be rewritten as system (8).

Consider the Lyapunov function $V = z^T P z$, where P is the positive definite matrix given in Lemma 1. Then, we can get

$$\begin{aligned} & V(k+1) - V(k) \\ & = z^T(k+1)Pz(k+1) - z^T(k)Pz(k) \\ & = \left(\left(I + \frac{1}{h}A - \frac{1}{h}BK \right) z(k) + \tilde{F}(k) \right)^T P \left(\left(I + \frac{1}{h}A - \frac{1}{h}BK \right) z(k) + \tilde{F}(k) \right) - z^T(k)Pz(k) \\ & = \frac{1}{h} z^T(k) \left((A - BK)^T P + P(A - BK) \right) z(k) + \frac{1}{h^2} z^T(k)(A - BK)^T P(A - BK) z(k) \\ & \quad + 2z^T(k) \left(I + \frac{1}{h}A - \frac{1}{h}BK \right)^T P\tilde{F}(k) + \tilde{F}^T(k)P\tilde{F}(k) \\ & \leq -\alpha \frac{1}{h} z^T(k)Pz(k) + \beta_1 \frac{1}{h^2} z^T(k)Pz(k) + 2\|P\| \left(\|I\| + \|A - BK\| \right) \|z(k)\| \|\tilde{F}(k)\| + \|P\| \|\tilde{F}(k)\|^2, \end{aligned} \tag{9}$$

where β_1 is a constant satisfying $(A - BK)^T P(A - BK) \leq \beta_1 P$, $\beta_1 > 0$.

Since $u_{\max}$ is known, and Assumption 1* holds with $\overline{\phi}$ being known. Then, when $h \geq 1$, it holds

$$\left| \frac{1}{h^{n+1-i}} f_i(x(k), u(k)) \right| \leq \frac{\overline{\phi}}{h^2} \left(|z_{i+2}(k)| + \cdots + |z_n(k)| + |Kz(k)| \right) \leq \frac{1}{h^2} \overline{\phi} \left(\sqrt{n} + \|K\| \right) \|z(k)\|,$$

for $i = 1, 2, \ldots, n-1$ and $f_n(x(k), u(k)) = 0$. Thus, we can get

$$\left\| \tilde{F}(k) \right\| \leq \frac{1}{h^2} \overline{\phi} \left(n + \|K\| \sqrt{n} \right) \|z(k)\|.$$

Substituting the above inequality into (9) can get

$$V(k+1) - V(k) \leq -\left(\alpha h - \beta_1 - \beta_2 \overline{\phi} - \beta_3 \overline{\phi}^2 \right) \frac{1}{h^2} V(k),$$

where $\beta_2 = 2 \frac{\|P\|}{\lambda_{\min}(P)} \left(\|I\| + \|A - BK\| \right) \left(n + \|K\| \sqrt{n} \right)$, and $\beta_3 = \frac{\|P\|}{\lambda_{\min}(P)} \left(n + \|K\| \sqrt{n} \right)^2$.

Thus, by choosing $h' = 2 \left(\beta_1 + \beta_2 \overline{\phi} + \beta_3 \overline{\phi}^2 \right) \frac{1}{\alpha}$, we can calculate to get

$$V(k+1) - V(k) \leq -\alpha \frac{1}{2h} V(k). \tag{10}$$

Then, it holds

$$V(k) \leq \left(1 - \alpha \frac{1}{2h} \right)^k V(0). \tag{11}$$

Now, we need to design h such that $|v(k)| \leq u_{\max}$. Let

$$h_{\min} \geq \max \left\{ h', r_{\Omega} \|K\| \frac{1}{u_{\max}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right\},$$

where r_Ω is the constant satisfying $\|x\| \le r_\Omega$ for all $x \in \Omega$. The constants $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are, respectively, the maximum and minimum eigenvalues of matrix P.

Because $|v(k)| \le \|K\|\,\|z(k)\|$, and $\|z(0)\| \le \frac{1}{h}\|x(0)\|$, we can get $|v(0)| \le u_{\max}$. Thus, it holds $V(1) \le V(0)$, which means

$$\|z(1)\| \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|z(0)\| \le \frac{u_{\max}}{\|K\|}.$$

Then $|v(1)| \le u_{\max}$. We repeat this process, and get $|v(k)| \le u_{\max}$ for any k.

Back to (11), we can get

$$\lim_{k\to+\infty} V(k) = 0.$$

Since h is a constant, we can get the convergence of $x(k)$. This ends the proof. ■

Remark 4. The peaking phenomenon can happen in the study of DFNSs. It is noticed that the relation between $x(k)$ and $V(k)$ can be expressed as

$$\frac{1}{h^{2n}}\lambda_{\min}(P)\|x(k)\|^2 \le V(k) \le \frac{1}{h^2}\lambda_{\max}(P)\|x(k)\|^2.$$

From (11), we can get

$$\|x(k)\|^2 \le h^{2n-2}\left(1 - \alpha_1\frac{1}{2h}\right)^k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(0)\|^2.$$

Thus, for a constant $h \ge h_{\min}$, the convergence $\|x(1)\|^2 \le \|x(0)\|^2$ may not be established. But when k is large enough, such as

$$k \ge \left\lceil \frac{\ln\left(\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}\frac{1}{h^{2n-2}}\right)}{\ln\left(1 - \alpha\frac{1}{2h}\right)} \right\rceil + 1,$$

condition $\|x(k)\|^2 \le \|x(0)\|^2$ can be guaranteed. This relation will be utilized in the next section.

4 | GLOBAL STABILIZATION WITH UNKNOWN SATURATION

When the saturation constrain $u_{\max}$ and the initial condition Ω are unknown, the constant h cannot be determined in Theorem 1. We try to determine the parameter h through data-driven scheme. The control (2) is improved as

$$v(k) = -k_1\frac{x_1(k)}{h^n(k)} - k_2\frac{x_2(k)}{h^{n-1}(k)} - \cdots - k_n\frac{x_n(k)}{h(k)}, \tag{12}$$

where k_1 to k_n are the same elements given in (2), and the dynamic parameter $h(k) \ge 1$ is to be designed.

Under the candidate control (12), the closed-loop system is turned into

$$x(k+1) = (I + A)x(k) - B\text{sat}\left(K\left(h(k)\right)x(k)\right) + F(k), \tag{13}$$

where $K(h(k)) = \left[k_1\frac{1}{h^n(k)}, k_2\frac{1}{h^{n-1}(k)}, \ \dots\ , k_n\frac{1}{h(k)}\right]$.

For the system state $x(k)$, we have the following estimation.

Theorem 2. *Supposed that Assumption 1* is satisfied. There exists constants ρ, τ such that the state of system (13) satisfies*

$$\|x(k+1)\| \le \rho\,\|x(k)\| + \tau. \tag{14}$$

Proof. From (13), it holds

$$\|x(k+1)\| \leq \|(I+A)\| \, \|x(k)\| + \|B\| \, u_{\max} + \|F(k)\| \, .$$

From Assumption 1*, we can get

$$|f_i(\cdot)| \leq \overline{\phi}\,(|x_{i+2}| + \cdots + |x_{n+1}| + u_{\max}) \leq \overline{\phi}\sqrt{n}\,\|x\| + \overline{\phi}u_{\max}$$

for $i = 1, 2, \ldots, n-1$ and $f_n = 0$. Then, we can achieve

$$\|F(k)\| = \sqrt{f_1^2 + f_2^2 + \cdots + f_n^2} \leq n\overline{\phi}\|x(k)\| + \overline{\phi}\sqrt{n}u_{\max}.$$

Thus, it holds

$$\|x(k+1)\| \leq \left(\|(I+A)\| + n\overline{\phi}\right)\|x(k)\| + \|B\| \, u_{\max} + \overline{\phi}\sqrt{n}u_{\max}.$$

Noticed that constants $\overline{\phi}$ and $u_{\max}$ may be unknown, but we know there are constants ρ, τ meeting (14), which ends the proof. ■

Let

$$z_1(k) = \frac{1}{h^n(k)}x_1(k), \ \ldots, z_n(k) = \frac{1}{h(k)}x_n(k),$$

and in this new set of coordinates $z = [z_1, z_2, \ldots, z_n]^T$, one can show that the input (12) is expressed as

$$v(k) = -Kz(k).$$

The estimation of the input can be given as

$$|v(k)| \leq \|K\|\|z(k)\| \leq \frac{1}{h(k)}\|K\|\|x(k)\|. \tag{15}$$

To make the input $v(t)$ pass the saturation constrain, we desire the dynamic gain $h(k)$ is larger than the state $\|x(k)\|$. And after the input $v(t)$ pass the saturation constrain $|v(k)| \leq u_{\max}$, we hope the dynamic gain $h(k)$ to be a large constant, which has been shown in Theorem 1 to stabilize the DFNS (1). Thus, by choosing the rate $\gamma > 0$, the parameter $h(k)$ can be updated through the following policy

$$h(k+1) = h(k) + \gamma \max\left\{\|x(k)\|^2, \ 1\right\}.$$

Denoting $H(k) = \mathrm{diag}\left\{\frac{1}{h^n(k)}, \ldots, \frac{1}{h(k)}\right\}$, and let

$$V_i^j = x^T(i)H(j)PH(j)x(i), \quad \text{for } \ i,j = 0, 1, 2, 3, \ldots .$$

It is noticing that the $h(k)$ is accepted at k, which means that the evaluation (10) can be achieved. That is

$$V_{k+1}^k \leq \left(1 - \alpha\frac{1}{2h(k)}\right)V_k^k.$$

Thus, the updating policy is summarized in Algorithm 1.

Theorem 3. *Supposed that Assumption 1* is satisfied. Consider system (1) under control (12). If the dynamic parameter $h(k)$ is updated through the Algorithm 1, the system state $x(k)$ satisfies*

$$\lim_{k\to+\infty} \|x(k)\| = 0, \tag{16}$$

and $h(k)$ is converging to a finite constant.

Algorithm 1. Update h through performance V^*_{**}

Initialize:
$h(0) \geq 1, k = 0, \tau > 0.$
while True **do**
if $V^k_{k+1} \leq (1 - \alpha\frac{1}{2h(k)})V^k_k$ **then**
$h(k+1) = h(k)$
else
$h(k+1) = h(k) + \gamma \max\left\{\|x(k)\|^2,\ 1\right\}$
end if
$k \leftarrow k+1$
end while

Proof. If there exists a finite instant T such that $k \geq T$,

$$V^k_{k+1} \leq \left(1 - \frac{1}{2h(k)}\alpha_1\right) V^k_k,$$

and

$$V^{T-1}_T > \left(1 - \frac{1}{2h(T-1)}\alpha_1\right) V^{T-1}_{T-1}$$

holds. Then, from Algorithm 1, we can get

$$h(k) = h(T), \quad V^k_{k+1} \leq \left(1 - \frac{1}{2h(k)}\alpha_1\right) V^k_k.$$

It is easy to deduce into $V^{k+1}_k = V^k_k = V^T_k$, and we obtain

$$V^T_k \leq \left(1 - \frac{1}{2h(T)}\alpha_1\right)^{k-T} V^T_T,$$

which yields $\lim_{k\to+\infty} V^T_k = 0$. Thus, we can get (16), and $h(k)$ is converging to a constant $h(T)$.

Now, we prove that the finite instant T can be found for any saturating constrain $u_{\max}$. If $v(k)$ pass the saturating constrain, from Theorem 1, there exists a constant h' such that when $h(T) \geq h'$, we have $V^k_{k+1} \leq \left(1 - \frac{1}{2}\alpha h^{-1}(k)\right) V^k_k, k \geq T$.

Let us consider the saturating constrain $|u(v(k))| \leq u_{\max}, k \geq T$. From the estimation (15), to ensure the input $v(k)$ can pass the saturating constrain, we need

$$\frac{\|x(k)\|}{h(k)} \leq \frac{u_{\max}}{\|K\|}.$$

Since $h(k+1) = h(k)$, $V^k_{k+1} \leq \left(1 - 1/2\alpha h^{-1}(k)\right) V^k_k$, we consider the peaking phenomenon and get

$$|v(k)| \leq \frac{\|x(k)\|}{h(k)}\|K\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\|x(T)\|}{h(T)}\|K\|, \quad k \geq T.$$

Thus, our problem turns to show that there exists a finite index T to meet

$$h(T) \geq \max\left\{\|x(T)\|\|K\|\frac{1}{u_{\max}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad h'\right\}. \tag{17}$$

Algorithm 2. Update h through performance $\|x\|^2$

Initialize:
$h(0) \geq 1, k = 0, \gamma > 0.$
while True **do**
$m \leftarrow \left\lceil \frac{-(2n-1)\ln h(k)}{\ln\left(1-\alpha_1 \frac{1}{2h(k)}\right)} \right\rceil + 1$
for $i = 1, 2, \dots, m-1$ **do**
$h(k+i) = h(k)$
end for
if $\|x(k+m)\|^2 \leq (1 - \alpha \frac{1}{2h(k)})\|x(k+m-1)\|^2$ **then**
$h(k+m) = h(k)$
else
$h(k+m) = h(k) + \gamma \max\left\{\|x(k)\|^{m+1}, 1\right\}$
end if
$k \leftarrow k + m$
end while

From Theorem 2, there exists constants ρ, τ to satisfy

$$\|x(T)\| \leq \rho\|x(T-1)\| + \tau.$$

Let ω_0 be the max solution of the equation

$$\gamma\omega_0^2 = (\rho\omega_0 + \tau)\,\|K\|\frac{1}{u_{\max}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}},$$

and let $\omega \geq \max\left\{\sqrt{h_{\min}/\gamma}, \omega_0\right\}$. Consider the index T be the updating index. If $\|x(T-1)\| \geq \omega$, we have

$$h(T) \geq \gamma\|x(T-1)\|^2 \geq \max\left\{\|x(T)\|\|K\|\frac{1}{u_{\max}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad h'\right\},$$

which meets the desired condition (17).

Consider the case when $\|x(T-1)\| < \omega$, and we have

$$\max\left\{\|x(T)\|\|K\|\frac{1}{u_{\max}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad h'\right\} \leq \max\left\{(\rho\omega + \tau)\,\|K\|\frac{1}{u_{\max}}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad h'\right\},$$

which is a bounded constant. It is noticing that for each updating time k, it holds $h(k+1) \geq h(k) + \gamma$. Thus, for any positive constant, after updating finite times, we can always get $h(T)$ to satisfy (17).

Thus, there exists an index T such that (17) is achieved. Then,

$$V_{k+1}^k \leq \left(1 - \frac{1}{2h(k)}\alpha_1\right) V_k^k,$$

holds for $k \geq T$. This ensures the state convergence and the bounded of dynamic parameter $h(k)$. This ends the proof. ■

In Algorithm 1, there are some limitation on the performance V_{**}^*. On the one hand, it is not easy to compute V_{**}^*. For each index $k = 0, 1, \dots$, we need to compute both V_k^k and V_{k+1}^k. On the other hand, V_{**}^* is chosen based on the positive definite matrix P, which may not be optimizable from the user-specified perspective. Thus, we improve it as the performance $\|x\|^2$, which is summarized in Algorithm 2.

Theorem 4. *Supposed that Assumption 1* is satisfied. Consider system (1) under control (12). If the dynamic parameter $h(k)$ is updated through Algorithm 2, the system state $x(k)$ satisfies*

$$\lim_{k\to+\infty} \|x(k)\| = 0.$$

Proof. If there exists a finite instant T such that

$$h(k) = h(T), \quad h(T) > h(T-1), \quad k \geq T.$$

Then, we can get

$$\|x(m_T i + T)\|^2 \leq \left(1 - \alpha \frac{1}{2h(T)}\right) \|x(m_T(i-1) + T)\|^2,$$

where $i = 1, 2, \ldots$, and

$$m_T = \left\lceil \frac{-(2n-1)\ln h(T)}{\ln\left(1 - \alpha_1 \frac{1}{2h(T)}\right)} \right\rceil + 1.$$

Thus, it holds

$$\lim_{i\to+\infty} \|x(m_T i + T)\|^2 = 0.$$

From Theorem 2, we can get constants ρ, τ such that

$$\|x(j + m_T i + T)\| \leq \rho^j \|x(m_T i + T)\| + \sum_{s=0}^{j-1} \rho^s \tau,$$

holds for $j = 1, 2, \ldots, m_T$. The following analysis is similar to that in Theorem 3, which is omitted here. This ends the proof. ∎

Remark 5. We employ the quadratic term $\|x(k)\|^2$ to get the upper bounded of $\rho\|x(k)\| + \tau$ for the unknown constants ρ, τ. But, when $\|x(k)\|$ is larger enough, it will generate a larger parameter h, which is not benefit to the system performance. Thus, by choosing a small constant ϵ, we can utilize $\|x(k)\|^{1+\epsilon}$ to replace $\|x(k)\|^2$.

5 | SIMULATION

Next, we present an example to illustrate the effectiveness of our designed framework.

Example. Consider

$$\begin{aligned}
x_1(k+1) &= x_1(k) + p_1 x_2(k) + a_1 u^2(v(k)) \sin(x_2(k)), \\
x_2(k+1) &= x_2(k) + p_2 x_3(k) + a_2 u(v(k)) x_4(k), \\
x_3(k+1) &= x_3(k) + p_3 x_4(k) + a_3 u^2(v(k)), \\
x_4(k+1) &= x_4(k) + u(v(k)),
\end{aligned} \tag{18}$$

where $x = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$ is the system state, and $u(v) \in \mathbb{R}$ is the control input with $v \in \mathbb{R}$ being the designed input. We choose the positive constants p_1, p_2, p_3 as 0.05, 0.5, and 1.2. Let $a_1 = -0.5$, $a_2 = 2$, and $a_3 = -0.6$. It is verified that Assumption 1 holds with $\phi(u) = 2u$. Then, the low gain feedback control is designed as

$$u(k) = -2\frac{1}{h^4} x_1(k) - 4\frac{1}{h^3} x_2(k) - 6\frac{1}{h^2} x_3(k) - 4\frac{1}{h} x_4(k), \tag{19}$$

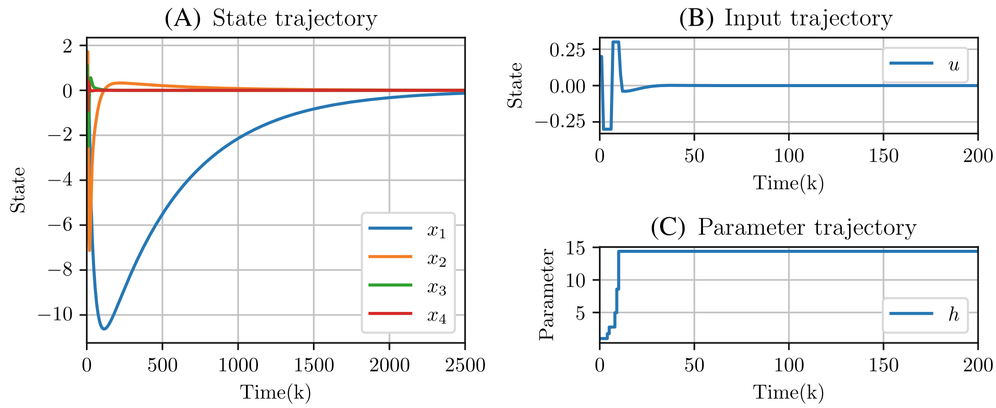

FIGURE 1 Simulation of system (18) and (19) under Algorithm 1

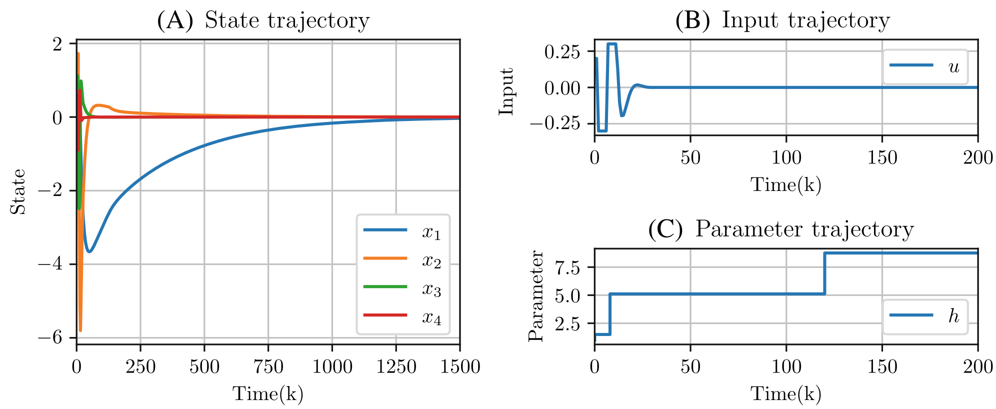

FIGURE 2 Simulation of system (18) and (19) under Algorithm 2

where h is a parameter. The considered initial state is assumed as $x(0) = [0.4, -0.2, -0.3, 0.4]^T$. We, respectively, employ Algorithms 1 and 2 to determine the parameter h, and verify its effectiveness.

Based on Algorithm 1, the state trajectory and input trajectory of system (18) and (19) are, respectively, shown in Figure 1. It is shown that the state is converging to zero, and the input is bounded by 0.3, which verifies the effectiveness of Algorithm 1. Similarly, Algorithm 2 is illustrated through noticing the state trajectory and input trajectory of system (18) and (19) in Figure 2. Thus, both Algorithms 1 and 2 can generate the stable closed-loop system, which verifies the effectiveness.

6 | CONCLUSION

The discrete-time low gain feedback control technology was introduced in this article, and the stabilizing control for DFNSs was designed respect to the unknown input saturation. Meanwhile, it is assumed that the nonlinear growth rate could be restricted by an input-depended function. By considering the low gain feedback control technology, we converted the design of the stabilizing control into the determination of a parameter. Then, we presented that the parameter exists to make the generated closed-loop system be stability. And then the system evolution was estimated to design the update policy for the parameter. We, respectively, considered the Lyapunov function and the Euclidean norm to evaluate the policy, and gave Algorithms 1 and 2 to calculate the parameter. The given numerical example illustrates the effectiveness of the designed controls.

CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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