Stabilization for a Class of Feedforward Nonlinear Systems via Pulsewidth-Modulated Controllers

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***Abstract*—This article is concerned with the stabilization problem for a class of feedforward nonlinear systems via pulsewidth-modulated (PWM) controllers. First, a state observer, which features a dynamic predictor and a jumping update, is designed such that the unavailable full system state can be accurately estimated. With the obtained state estimations, a reference signal is also constructed. Second, a three-level-state-dependent PWM controller is proposed, which switches among three prescribed states during each cycle and can be determined by two key variables, namely, the duty cycle and the sign of the pulse. It is then demonstrated that the two controller variables can be both expressed in the form of the reference signal, and the design of the controller variables depends on a key parameter that regulates the reference signal. Furthermore, it is shown that the stabilization problem can be cast into a selection problem of the key parameter. By resorting to the Lyapunov stability theory, rigorous stability analysis of the resulting closed-loop system is performed. Finally, a resonant circuit system for liquid level control and a numerical example are provided to validate the effectiveness of the proposed PWM control method.**

***Index Terms*—Duty cycle, feedforward nonlinear systems, hybrid observer, pulsewidth, pulsewidth-modulated (PWM) control, stabilization.**

I. Introduction

Benefiting from the development of networked control technologies, various shortcomings of analog-signal-based controllers, such as drifting of control signals with time, difficulties of signal adjusting, and different interferences, can be effectively overcome via digital control techniques. A typical feature of such a digital control technique is the ON–OFF switching of the actuator (or controller) [1], which prominently reduces the communication power resource between the actuator (or controller) and the plant. Such an ON–OFF characteristic is also beneficial to encode desired continuous-time information into digital bits, which aligns with nowadays wireless communication paradigms. However, it should be pointed out that the ON–OFF switching actuator (or controller) essentially represents a binary device whose dynamic characteristics cannot be described by linear models. Therefore, ground-breaking theoretical analysis and controller design are demanded to restrain the performance degeneration of digital control systems coming from such binary devices.

A typical ON–OFF control scheme is the pulsewidth-modulated (PWM) control. Specifically, in the context of PWM control, the time is partitioned into a series of equal cycles, and then the PWM control signal turns the actuator ON over a portion of each cycle and switches it OFF during the rest of the cycle. The PWM duty cycle thus describes the proportion of ON time to the regular interval or period of time and is regarded as a new control variable. Much research effort has therefore been devoted to determining the PWM duty cycle fulfilling the required control performance [2], [3], [4], [5]. So far, several attractive methods have been developed in the past few years, such as sliding mode control [6], [7], hybrid control [8], [9], and linear quadratic (LQ) optimal control [10], [11]. Nevertheless, it is noteworthy that the inherent nonlinearity of the physical systems is unfortunately neglected in these studies, which implies that the desired control performance may not be preserved in realistic scenarios as most practical control systems are complexly nonlinear.

Feedforward nonlinear systems, as a representative class of nonlinear systems, have been commonly employed to model some practical physical systems, such as inertia wheel pendulums [12], planar vertical takeoff and landing aircraft [13], and translational oscillation with a rotational actuator [14]. Such a feedforward nonlinear system has a triangular structure and hence is also referred to an upper-triangular system. Since the pioneering work of [15], the low-gain feedback control methods [16], [17], [18], [19], [20] and the nested saturation control methods [14], [21], [22], [23] have been greatly developed to achieve the design of desired feedback controllers for such nonlinear systems. However, it should be stressed that the applicability of those control methods relies on the smoothness of the controllers. As such, a zero order or zero holder is often required to keep the control signal constant during the implementation of the controllers, especially when quantization or sampled data are a concern. Notwithstanding, different from quantized or sampled-data controllers, a PWM controller has relatively limited control capability and only possesses two-level states (1 and -1) or three-level states ($1, 0,$ and -1). As a result, the following two essential challenging issues are identified for achieving the PWM controller design of feedforward nonlinear systems: 1) how to choose the state during the ON time and determine the ON portion of each cycle, and 2) how to deal with the switching effects resulted from the PWM controller on the concerned feedforward nonlinear system while still guaranteeing its closed-loop stability. To the best of authors' knowledge, the above two issues remain challenging in the PWM control literature, which motivates this study.

In this article, we develop a novel PWM controller design method that provides feasible solutions to the above two questions. The specific contributions are summarized as follows.

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1) *A hybrid observer*, featuring a dynamic predictor and a jumping update, is designed to estimate the full system state and provides a reference signal for the desired PWM controller.
2) *A delicate three-level-state-dependent PWM controller* is constructed and designed. In particular, the controller relies on two key control variables, namely, the duty cycle and the sign of the pulse, which can both be determined by the reference signal. It is further demonstrated that a key parameter h_k that dynamically adjusts the reference signal plays a key role in achieving the efficient PWM control and performing the stability analysis.
3) *Formal stability criteria in terms of global asymptotic stability and semiglobal asymptotic stability* are derived, respectively, to account for a dynamic parameter case of h_k and a constant parameter case of $h_k \equiv h$. Specifically, an equivalent discretization version of the closed-loop system is first derived to accommodate the switching effects resulted from the PWM controller and the influences from the resultant nonlinear residuals and approximation errors. The stabilization problem is then converted into a selection problem of the parameter h_k.

II. Problem Formulation

Consider a class of feedforward nonlinear systems that can be described by

$$\dot{x}(t) = Ax(t) + Bu(t) + F\left(x(t), u(t)\right) \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the system state, and $u(t) \in \mathbb{R}$ is the control input. The initial state is denoted as $x(t_0) \in \mathbb{R}^n$. Without loss of generality, the initial instant t_0 is assumed as 0, and the equilibrium is $x = 0$. Matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ satisfy

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

Furthermore, the function $F = (f_1, f_2, \ldots, f_n)^T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous with respect to each of its variables, and satisfies the following assumption.

Assumption 1: For any $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$, it holds that $|f_i(x, u)| \le \theta(|x_{i+2}| + \ldots + |x_n| + |u|)$, $i = 1, \ldots, n-2$, and $|f_{n-1}(u)| \le \theta|u|$, $f_n = 0$, where θ is a positive constant.

Remark 1: Assumption 1 has been commonly made in the existing literature, e.g., [20], [24], [25], and [26], aiming at solving the stabilization problem of feedforward nonlinear systems via enough control capability. However, different from these results, we will demonstrate that the feedforward nonlinear system (1) under Assumption 1 can be stabilized by a controller with only three-level states.

In this study, the above system is regulated by a discretized and sampled-data controller, which implies that only sampled system measurements are available for determining the desired control input. For this purpose, we let the sample sequence be $\{t_k\}_{k\ge 0}$, where $t_k = kT$ and $T > 0$ denotes a fixed sampling period. Then, the measured system output can be described by

$$y(t_k) = Cx\left(t_k\right) \tag{3}$$

where $C = (1, 0, \ldots, 0)$ is the measurement matrix.

Furthermore, we propose the following three-level-state-dependent PWM control law to stabilize the system (1):

$$u(t) = \begin{cases} ms_k, & t \in [t_k, t_k + \delta_k) \\ 0, & t \in [t_k + \delta_k, t_{k+1}) \end{cases} \tag{4}$$

where m is a prescribed positive constant representing the *amplitude* of the PWM signal, $s_k \in \{-1, 1\}$ represents the *sign* of the pulse during the kth sampling period, and δ_k is known as the *pulsewidth* during the kth sampling period. To make the PWM signal well defined, it is required that $0 \le \delta_k \le T$. Then, the *duty cycle* of the PWM is defined as

$$d_k = \frac{\delta_k}{T} \cdot 100\% \in [0\%, 100\%]$$

which describes the proportion of ON time to the regular interval T. In particular, when the control is OFF during kth sampling period, it has a duty cycle of $d_k = 0\%$. A low duty cycle corresponds to a low power of the control because the power is OFF for most of the time. Since the PWM control incorporates the constant control and the open-loop control, it has limited control capability.

It can be seen from the PWM control law (4) that the control signal $u(t)$ is determined by the duty cycle d_k and the sign s_k. Therefore, *the main objective of this article* is to design the two controller variables d_k and s_k such that under the proposed PWM control law (4), the equilibrium point $x = 0$ of the resulting closed-loop system become asymptotically stable.

III. Design of the Desired PWM Controller

This section focuses on the concrete design of controller variables d_k and s_k. To this end, a state observer is introduced first, based on which a reference signal v_k is skillfully derived.

A. Design of a Hybrid State Observer and a Reference Signal

Given that the system output $y(t)$ is only available at specific instants $\{t_k\}_{k\ge 0}$, we reconstruct the system full state via the following hybrid observer[1]:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad t \in [t_k, t_{k+1}) \tag{5a}$$

$$\hat{x}\left(t_{k+1}\right) = \hat{x}\left(t_{k+1}^-\right) + \frac{T}{h_k^{n+1}} H_k^{-1} L\left(C\hat{x}\left(t_{k+1}^-\right) - y(t_{k+1})\right) \tag{5b}$$

where $\hat{x}(t) \in \mathbb{R}^n$ denotes the state estimate with $\hat{x}(t_0)$ being an initial estimate, $L \in \mathbb{R}^n$ is the observer gain vectors to be determined, and $H_k = \operatorname{diag}\left\{\frac{1}{h_k^n}, \frac{1}{h_k^{n-1}}, \ldots, \frac{1}{h_k}\right\}$ with the dynamic parameter h_k being updated by

$$h_{k+1} = h_k \max\left\{\gamma\|H_k\hat{x}(t_{k+1})\|, 1\right\}. \tag{6}$$

Here, γ is a constant to be specified later, and the initial value h_0 satisfying $h_0 \ge 1$ also will be specified later.

Based on the observer above, a reference signal v_k is constructed as follows:

$$v_k = KH_k\hat{x}\left(t_k\right) \tag{7}$$

where $K^T \in \mathbb{R}^n$ is a gain matrix to be determined later. This reference signal v_k will be utilized latter to design the PWM controller variables.

Let

$$\mathscr{A} = \begin{pmatrix} A + LC & 0 \\ -LC & A + BK \end{pmatrix}, \quad \mathscr{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

with $D = \operatorname{diag}\{n, n-1, \ldots, 1\}$. Then, following the similar algorithms in [27] and [28], we choose the vectors $K^T \in \mathbb{R}^n$ and $L \in \mathbb{R}^n$,

[1]The solution $\hat{x}(\cdot)$ is a right-continuous function. Given a right-continuous function $\phi : \mathbb{R} \to \mathbb{R}^n$, the notation $\phi(t^-)$ stands for $\lim_{h\to 0^-} \phi(t + h)$.

and positive definite matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ such that

$$P_1(A+BK)+(A+BK)^T P_1 \le -I, P_1 D + DP_1 \ge \alpha_1 P_1$$

$$P_2(A+LC)+(A+LC)^T P_2 \le -I, P_2 D + DP_2 \ge \alpha_2 P_2$$

hold, where α_1 and α_2 are two positive constants. In what follows, by choosing positive definite matrix $P = \mathrm{diag}\{\mu P_2, P_1\}$ with $\mu = \|P_1 LC\|^2 + 1$, one has

$$\mathscr{A}^T P + P\mathscr{A} \le -\varrho I, \quad P\mathscr{D} + \mathscr{D}P \ge \sigma P \tag{8}$$

where $\varrho \le 1$, and $\sigma \le \min\{\mu\alpha_2, \alpha_1\}$ are positive constants.

B. Design of the PWM Controller Variables

Based on the reference signal v_k, the duty cycle d_k and the sign s_k of the pulse are designed as follows:

$$s_k = \mathrm{sign}\,(v_k)\,, \quad d_k = \begin{cases} 100\% & |v_k| > m \\ \frac{|v_k|}{m}\cdot 100\% & |v_k| \le m \end{cases} \tag{9}$$

where m is given in (4). It can be seen from (9) that $|v_k| > m$ leads to the case that the control is always ON. To effectively regulate the duty cycle d_k, it is desired that the bound of the signal $|v_k|$ should become smaller than m during most periods. Recalling (6) and (7), the design of the reference signal v_k relies on the dynamic parameter h_k and thus the scalar γ. Hence, we next provide a criterion on the selection of parameter γ.

From the dynamics (6), one has that $h_{k+1} \ge h_k \ge \cdots \ge h_0 \ge 1$. Then, it can be deduced from (5b) and (6) that, when $k \ge 0$, we obtained that $\frac{h_{k+1}}{h_k} \ge \gamma \|H_k \hat{x}(t_{k+1})\|$. On the other hand, one has that $|v_{k+1}| = |KH_{k+1}\hat{x}(t_{k+1})| \le \frac{h_k}{h_{k+1}}\|K\|\|H_k\hat{x}(t_{k+1})\|$, which means $|v_{k+1}| \le \|K\|/\gamma$. Therefore, by choosing

$$\gamma \ge \frac{\|K\|}{m} \tag{10}$$

one can ensure that $|v_{k+1}| \le m$. In this case, we obtain

$$v_{k+1} = m s_{k+1} d_{k+1}. \tag{11}$$

Remark 2: It is shown that the dynamic parameter h_k plays an important role in adjusting the reference signal v_k. More concretely, given the prescribed PWM amplitude m and the determined gain matrix K, a selection of the parameter γ satisfying (10) ensures that $|v_k| \le m$. Thus, the duty cycle d_k of PWM can be effectively regulated. On the other hand, it is clear from (7) that the upper bound of v_0 depends on the initial observer state $\hat{x}(t_0)$, which may be larger than m. Thus, the condition (11) may fail for v_0. However, this does not affect our design since the finite time escape does not happen in the period $t \in [t_0, t_1)$.

We next reveal the explicit relationship between the PWM control input $u(t)$ and the signal v_k. On one hand, one has the following integration during the period $[t_{k+1}, t_{k+2})$:

$$\frac{1}{T}\int_{t_{k+1}}^{t_{k+2}} u(t)dt = \frac{1}{T}\int_{t_{k+1}}^{t_{k+1}+\delta_{k+1}} m s_{k+1} dt = \frac{1}{T} m s_{k+1}\delta_{k+1}.$$

On the other hand, it follows from (11) that:

$$\frac{v_{k+1}}{m} = s_{k+1} d_{k+1} = s_{k+1}\frac{\delta_{k+1}}{T}$$

when $|v_k| \le m$, which results in that

$$\frac{1}{T}\int_{t_{k+1}}^{t_{k+2}} u(t)dt = v_{k+1}$$

which means that the average power of $u(t)$ during the $(k+1)$th sampling period $[t_{k+1}, t_{k+2})$ is v_{k+1}.

Before ending this section, we present an algorithm, i.e., Algorithm 1, to outline some key design steps and the implementation of the proposed PWM controller.

Algorithm 1: Implementation of PWM Controller (4).

`/* Offline Design */`
Given the matrices A, B, C in (2), (3), select $K^T \in \mathbb{R}^n$, $L \in \mathbb{R}^n$ and $P \in \mathbb{R}^{2n\times 2n}$ such that (8) holds;
For the obtained K, choose a suitable γ satisfying (10);
Choose initial h_0 satisfying (21) or constant h satisfying (26);
Initialize $\hat{x}(t_0)$;
`/* Online Design */`
while $t \in [t_k, t_{k+1})$ **do**
  if $k \neq 0$ **then**
    Update the state $\hat{x}(t_k)$ through (5b);
    if *Case_1* **then**
      `/* Case_1: Dynamic` h_k `*/`
      Update h_k according to (6);
    else
      `/* Case_2: Constant` $h_k \equiv h$ `*/`
      Select a constant $h_k \equiv h$;
    end
  end
  Calculate the reference signal v_k according to (7);
  Determine s_k and d_k based on (9);
  Implement $u(t) = ms_k$, $t \in [t_k, t_k + \delta_k)$ on system (1);
  Predict $\hat{x}(t_{k+1}^-)$ at the next jumping instant by (5a);
  Update the sampling index $k = k+1$.
end

IV. Stability Analysis

In this section, we first present an equivalent discretization version of the closed-loop system and then derive formal stability analysis criteria related to two cases of a dynamic parameter h_k and a constant parameter $h_k \equiv h$.

A. Equivalent Discretization of the Closed-Loop System

Note that the parameter h_k, either dynamic or constant, does not affect the derivation procedure to obtain the equivalent discretization. Thus, we only consider the dynamic parameter case here. Denote $e(t) = x(t) - \hat{x}(t)$ as the estimation error. It follows from (1) and (5a) that:

$$\begin{cases} \dot{e}(t) = Ae(t) + F\left(x(t), u(t)\right), & t \in [t_k, t_{k+1}) \\ e\left(t_{k+1}\right) = \left(I + \frac{T}{h_k^{n+1}} H_k^{-1} LC\right) e\left(t_{k+1}^-\right). \end{cases} \tag{12}$$

For convenience of analysis, let us denote $F(x(t), u(t))$ as F_{xu}. One has

$$e\left(t_{k+1}^-\right) = \exp(AT)e\left(t_k\right) + \mathcal{F}_k$$

where $\mathcal{F}_k = \int_{t_k}^{t_{k+1}} \exp(A(t_{k+1}-s))F_{xu} ds$. Keeping the update at t_{k+1} in mind, one obtains

$$e\left(t_{k+1}\right) = \left(I + \frac{T}{h_k^{n+1}} H_k^{-1} LC\right)\exp(AT)e\left(t_k\right) + \mathcal{F}_{1\,k}$$

where $\mathcal{F}_{1\,k} = (I + \frac{T}{h_k^{n+1}} H_k^{-1} LC)\mathcal{F}_k$.

Let $\alpha_k = T/h_k$. Due to $H_k \exp(AT) = \exp(A\alpha_k)H_k$ and $C = h_k^n C H_k$, one gets

$$H_k e\left(t_{k+1}\right) = \Phi_1\left(\alpha_k\right) H_k e\left(t_k\right) + H_k \mathcal{F}_{1\,k}$$

where $\Phi_1(\alpha_k) = (I + \alpha_k LC)\exp(A\alpha_k)$.

On the other hand, substituting the PWM control input (4) into the observer dynamic (5a) leads to

$$\hat{x}\left(t_{k+1}\right) = \exp(AT)\hat{x}\left(t_k\right) - \frac{T}{h_k^{n+1}} H_k^{-1} LCe\left(t_{k+1}^-\right) + \int_0^{\delta_k} \exp\left(A\left(T-s\right)\right) Bds \cdot ms_k$$

$$= \exp(AT)\hat{x}(t_k) - \frac{T}{h_k^{n+1}} H_k^{-1} LC \exp(AT) e(t_k)$$
$$+ \int_0^{\delta_k} \exp(A(T-s)) B ds \cdot m s_k - \mathcal{F}_{2k}$$

where $\mathcal{F}_{2k} = \frac{T}{h_k^{n+1}} H_k^{-1} LC \mathcal{F}_k$. Then, one has

$$\begin{aligned} &H_k \hat{x}(t_{k+1}) \\ &= \exp(A\alpha_k) H_k \hat{x}(t_k) \\ &\quad - \alpha_k LC \exp(A\alpha_k) H_k e(t_k) - H_k \mathcal{F}_{2k} \\ &\quad + \int_0^{\delta_k} \exp\left(\frac{1}{h_k} A(T-s)\right) \frac{1}{h_k} B ds \cdot m s_k. \end{aligned} \tag{13}$$

Letting $\Phi_2(\alpha_k) = \exp(A\alpha_k)(I + \alpha_k BK)$ and

$$\begin{aligned} \mathcal{F}_{3k} = &\int_0^{\delta_k} \exp\left(\frac{1}{h_k} A(T-s)\right) \frac{1}{h_k} B ds \cdot m s_k \\ &- \alpha_k \exp(A\alpha_k) BK H_k \hat{x}(t_k) \end{aligned} \tag{14}$$

be the resulting effects from the PWM approximation error during the period $[t_k, t_{k+1})$. Then, the above dynamics (13) can be further rewritten as

$$\begin{aligned} H_k \hat{x}(t_{k+1}) = &\Phi_2(\alpha_k) H_k \hat{x}(t_k) \\ &- \alpha_k LC \exp(A\alpha_k) H_k e(t_k) \\ &- H_k \mathcal{F}_{2k} + \mathcal{F}_{3k}. \end{aligned}$$

Now, introducing the notations

$$\mathscr{H}_k = \operatorname{diag}\{H_k, H_k\},\ Z_k = \begin{pmatrix} e^T(t_k) & \hat{x}^T(t_k) \end{pmatrix}^T$$
$$\mathscr{L}(\alpha_k) = \begin{pmatrix} \Phi_1(\alpha_k) & 0 \\ -\alpha_k LC \exp(A\alpha_k) & \Phi_2(\alpha_k) \end{pmatrix}$$
$$\mathscr{F}_{1k} = \begin{pmatrix} \mathcal{F}_{1k}^T H_k^T & -\mathcal{F}_{2k}^T H_k^T \end{pmatrix}^T$$
$$\mathscr{F}_{2k} = \begin{pmatrix} 0 & \mathcal{F}_{3k}^T \end{pmatrix}^T \tag{15}$$

one has the following discrete-time closed-loop system in a compact form:

$$\mathscr{H}_k Z_{k+1} = \mathscr{L}(\alpha_k) \mathscr{H}_k Z_k + \mathscr{F}_{1k} + \mathscr{F}_{2k}. \tag{16}$$

For the above discrete-time nonlinear system under the condition $|v_k| \le m$, we present the following propositions whose proofs are given in Appendices A–C.

Proposition 1: There exists a positive constant ϑ_0 such that

$$\max_{t \in [t_k, t_{k+1})} \{\|H_k e(t)\| + \|H_k \hat{x}(t)\|\} \le \vartheta_0 \|\mathscr{H}_k Z_k\|. \tag{17}$$

Proposition 2: For the nonlinear term $\mathscr{F}_{1k}$ in the discrete-time system (16), there is a positive constant ϑ such that

$$\|\mathscr{F}_{1k}\| \le \alpha_k^2 \vartheta \|\mathscr{H}_k Z_k\|. \tag{18}$$

Proposition 3: The approximation error term $\mathscr{F}_{2k}$ in the discrete-time system (16) satisfies

$$\|\mathscr{F}_{2k}\| \le \frac{1}{2}\sqrt{n-1}\alpha_k^2 d_k \exp(\bar{\alpha}\|A\|)\|K\| \|\mathscr{H}_k Z_k\|$$

where $\bar{\alpha}$ is the upper bound of α_k.

Remark 3: Two prominent features of the derived equivalent closed-loop system in (16) are highlighted as follows. 1) The term $\mathcal{F}_{3k}$ as specified in (14) quantifies the effects caused by the PWM approximation error. Under the designed PWM controller, the upper bound of such an error is dependent on both the duty cycle d_k and the parameter h_k^2 included in α_k^2 (i.e., $\alpha_k^2 = \frac{T^2}{h_k^2}$). 2) The nonlinear term $\mathscr{F}_{2k}$ is also characterized by the parameter h_k^2. A conservative design is to choose an appropriate initial value h_0 or a suitable constant $h_k \equiv h$. In either case, we will next provide a detailed selection of the value of h_0 or h to regulate both the nonlinear term $\mathscr{F}_{1k}$ and the PWM approximation error $\mathscr{F}_{2k}$.

B. Stability Analysis

Before analyzing the stability, let us present the following lemmas, whose proofs are given in Appendices D and E.

Lemma 1: Let $\mathscr{L}(\alpha)$ be defined in (15), and the matrix P be the solution of the inequality (8) with given positive constants σ and ϱ. Then, there exists a constant $\bar{\alpha}$ such that, for any $\alpha \in (0, \bar{\alpha}]$, the following inequalities:

$$\mathscr{L}^T(\alpha) P \mathscr{L}(\alpha) - P \le -\alpha\frac{\varrho}{2} I, \quad \|\mathscr{L}(\alpha)\| \le \rho \tag{19}$$

hold, where ρ is a prescribed positive constant.

Lemma 2: Let the matrix P be the solution of the matrix inequalities (8) for given positive constants ϱ and σ. Then,

$$\bar{\mathcal{H}} P \bar{\mathcal{H}} \le \hbar^\sigma P \tag{20}$$

holds, where $\bar{\mathcal{H}} = \operatorname{diag}\{\hbar^n, \hbar^{n-1}, \ldots, \hbar, \hbar^n, \hbar^{n-1}, \ldots, \hbar\}$ with any $\hbar \in (0, 1]$.

In the case of a dynamic parameter h_k, the following stability analysis result is presented.

Theorem 1: Suppose that Assumption 1 holds, and let the matrix P be the solution to (8) for given vectors K and L, and positive constants ϱ and σ. Then, the equilibrium point $x = 0$ of system (1) is globally asymptotically stabilized through the PWM controller (4) and (9) with the observer (5a) and the dynamic parameter h_k (6) if the initial value of h_k satisfies

$$h_0 \ge \max\left\{\frac{4T(2\rho\Theta\|P\| + \Theta^2\|P\|)}{\varrho}, T, 1\right\} \tag{21}$$

where $\Theta = \vartheta + \frac{1}{2}\sqrt{n-1}\exp(\|A\|)\|K\|$ with ϑ, ρ, and $\bar{\alpha}$ being specified in Proposition 2 and Lemma 1, respectively.

Proof: Considering (21) and $h_k \ge h_0$, one has

$$\alpha_k \le \min\left\{\frac{\varrho}{4(2\rho\Theta\|P\| + \Theta^2\|P\|)}, 1\right\}. \tag{22}$$

For the closed-loop system (16), we can rewrite the nonlinear term and the error term together as

$$\mathscr{H}_k Z_{k+1} = \mathscr{L}(\alpha_k)\mathscr{H}_k Z_k + \mathscr{F}_k$$

where $\mathscr{F}_k = \mathscr{F}_{1k} + \mathscr{F}_{2k}$.

Following Lemma 1, one has

$$\begin{aligned} &Z_{k+1}^T \mathscr{H}_k P \mathscr{H}_k Z_{k+1} - Z_k^T \mathscr{H}_k P \mathscr{H}_k Z_k \\ &\le (\mathscr{L}(\alpha_k)\mathscr{H}_k Z_k + \mathscr{F}_k)^T P (\mathscr{L}(\alpha_k)\mathscr{H}_k Z_k + \mathscr{F}_k) \\ &\quad - Z_k^T \mathscr{H}_k P \mathscr{H}_k Z_k \\ &\le -\frac{\varrho\alpha_k}{2}\|\mathscr{H}_k Z_k\|^2 + 2Z_k^T \mathscr{H}_k \mathscr{L}(\alpha_k) P \mathscr{F}_k + \mathscr{F}_k^T P \mathscr{F}_k. \end{aligned}$$

Since $\alpha_k \le 1$, one can employ Propositions 2 and 3 to get

$$\|\mathscr{F}_k\| \le \alpha_k^2 \Theta \|\mathscr{H}_k Z_k\|.$$

Then, we further deduce into

$$\begin{aligned} &Z_{k+1}^T \mathscr{H}_k P \mathscr{H}_k Z_{k+1} - Z_k^T \mathscr{H}_k P \mathscr{H}_k Z_k \\ &\le -\left(\frac{\varrho}{2} - 2\Theta\rho\|P\|\alpha_k - \Theta^2\|P\|\alpha_k^3\right)\alpha_k \|\mathscr{H}_k Z_k\|^2. \end{aligned} \tag{23}$$

Substituting (22) into (23) yields

$$Z_{k+1}^T \mathscr{H}_k P \mathscr{H}_k Z_{k+1} \le \left(1 - \frac{\varrho}{4}\alpha_k \frac{1}{\lambda_{\max}(P)}\right) Z_k^T \mathscr{H}_k P \mathscr{H}_k Z_k \tag{24}$$

with $\lambda_{\max}(P)$ being the maximum eigenvalue of matrix P.

Select the Lyapunov candidate $V_k = Z_k^T \mathscr{H}_k P \mathscr{H}_k Z_k$. Following Lemma 2, we can get

$$\begin{aligned} V_{k+1} &= Z_{k+1}^T \mathscr{H}_k \mathscr{H}_k^{-1} \mathscr{H}_{k+1} P \mathscr{H}_{k+1} \mathscr{H}_k^{-1} \mathscr{H}_k Z_{k+1} \\ &\le \left(\frac{h_k}{h_{k+1}}\right)^{\sigma} Z_{k+1}^T \mathscr{H}_k P \mathscr{H}_k Z_{k+1} \\ &\le \left(\frac{h_k}{h_{k+1}}\right)^{\sigma} \left(1 - \frac{\varrho}{4}\alpha_k \frac{1}{\lambda_{\max}(P)}\right) V_k. \end{aligned} \tag{25}$$

We conclude that $h_k^{\sigma} V_k$ is bounded, and it holds $V_k \le V_0$. Since V_k is dependent on the matrix $\mathscr{H}_k$, we need to show that $\mathscr{H}_k$ is bounded. In other words, one needs to discuss the upper bound of h_k. To this end, we have

$$\|H_k \hat{x}(t_{k+1})\|^2 \le \frac{1}{\lambda_{\min}(P)} V_k \le \frac{1}{\lambda_{\min}(P)} V_0.$$

Because of (6), it is not difficult to find that

$$h_k \le h_0 \gamma \left(\frac{1}{\lambda_{\min}(P)} V_0\right)^{\frac{1}{2}}.$$

Finally, taking the relationship between $\mathscr{H}_k$ and h_k into account, one concludes the convergence of Z_k from (25). Therefore, the system (1) is globally asymptotically stabilized via the designed PWM controller, which ends the proof. ■

Remark 4: Theorem 1 provides a selection criterion for the initial value of the dynamic parameter h_k such that the closed-loop system under the PWM controller is globally asymptotically stable. Specifically, the selection of h_0 is greatly affected by two scalars: the nonlinearity-related constant ϑ and the system order n. As such, a relatively large h_0 could be selected to accommodate the large nonlinearity effect (due to a large ϑ) and increasing system scale (due to a large n). This is different from the existing dynamic gain feedback control studies in [16], [18], and [24], where the dynamic gain parameter itself is updated to deal with the nonlinearities.

Remark 5: It is noted that aperiodic sampling is allowed in our design. In this case, there are two constants $T_{\min}$ and $T_{\max}$ such that the sampling size $t_{k+1} - t_k$ satisfies $T_{\min} \le t_{k+1} - t_k \le T_{\max}$. Then, the initial value of parameter h can be designed based on the constants $T_{\min}$ and $T_{\max}$.

Theorem 1 is derived on the basis of a dynamic h_k that ensures $|v_{k+1}| < m$ when the initial state $x(t_0)$ of the system is not known a priori. However, when the initial system state is known, an alternative design solution is to select a constant parameter $h_k \equiv h$ to ensure the condition $|v_k| \le m,\ k \ge 0$, which is presented in the following theorem.

Theorem 2: Suppose that Assumption 1 holds, and let the matrix P be the solution to (8) for given vectors K and L, and positive constants ϱ and σ. Then, for any initial state $x(t_0)$, the equilibrium point $x = 0$ of system (1) is semiglobally asymptotically stabilized through the PWM controller (4), (9) based on the observer (5a) with constant parameter h if it satisfies

$$h \ge \max\left\{\frac{4\,T\,(2\Theta\rho\,\|P\| + \Theta^2\,\|P\|)}{\varrho},\ T,\ 1,\ \Delta\right\} \tag{26}$$

where $\Theta = \vartheta + \frac{1}{2}\sqrt{n-1}\exp(\bar{\alpha}\|A\|)\|K\|$ with $\vartheta, \rho, \bar{\alpha}$ being specified in Proposition 2 and Lemma 1, respectively, and Δ being a constant that satisfies

$$\Delta \ge \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\|K\|}{m} \|Z_0\|.$$

Here, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ stand for the maximum and minimum eigenvalues, and $Z_0 = ((x(t_0) - \hat{x}(t_0))^T\ \hat{x}^T(t_0))^T$.

Proof: The purpose is to find a suitable constant h such that $|v_k| \le m$ is always true. To this end, the mathematical induction is employed. First, it holds that

$$|v_0| \le \|K\|\|H_0\|\|\hat{x}(t_0)\| \le \frac{1}{h}\|K\|\|\hat{x}(t_0)\| \le m.$$

Then assume that $|v_i|$ is smaller than m for any $i = 0, 1, \ldots, k$. In what follows, one needs to verify $|v_{k+1}| \le m$.

Similar to (24) in the proof of Theorem 1, we can also get

$$Z_{i+1}^T \mathscr{H} P \mathscr{H} Z_{i+1} \le \left(1 - \frac{\varrho}{4}\alpha \frac{1}{\lambda_{\max}(P)}\right) Z_i^T \mathscr{H} P \mathscr{H} Z_i$$

where $\alpha = \frac{T}{h}$ and $\mathscr{H} = \mathrm{diag}\{H, H\}$.

Let $V_k = Z_k^T \mathscr{H} P \mathscr{H} Z_k$, and one have

$$V_{i+1} \le \left(1 - \frac{\varrho}{4}\frac{T}{h}\frac{1}{\lambda_{\max}(P)}\right)^i V_0, \quad i = 0, 1, \ldots, k \tag{27}$$

which indicates

$$\|\mathscr{H} Z_{k+1}\|^2 \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \left(1 - \frac{\varrho}{4}\frac{T}{h}\frac{1}{\lambda_{\max}(P)}\right)^k \|\mathscr{H} Z_0\|^2.$$

Since h satisfies $\|\mathscr{H} Z_0\| \le \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \frac{m}{\|K\|}$, one has

$$\|\mathscr{H} Z_{k+1}\| \le \frac{m}{\|K\|}.$$

Therefore, it is not difficult to get $|v_{k+1}| \le m$. In summary, for any k, $|v_k| \le m$ is always true, and (27) is always holding. Thus, the closed-loop system is asymptotically stable. This ends the proof. ■

Remark 6: Theorems 1 and 2 provide sufficient conditions for the parameters h_0 and h, respectively, to achieve the asymptotic stabilization. If one chooses these parameters as large constants, the system performance would be degraded, and has a slow converging rate and a small duty cycle. To get a better performance, both the sampling size T and the parameter h_0 (or h) should be considered in the design.

Remark 7: To achieve output feedback stabilization of nonlinear feedforward systems (1), two typical control methods have been available in the existing literature: 1) sampled-data or quantized control [29], [30], where the controller switches among multiple levels, and 2) continuous-time control [16], [20], [24]. As shown in Theorems 1 and 2, we provide a thorough solution to switching PWM control design among three-level states. In addition, Theorem 1 is complimentary to the work [1], which is concerned about PWM control design for linear systems, as our result is dedicated to a class of nonlinear systems under Assumption 1.

V. Simulation

In this section, two examples are employed to illustrate the effectiveness of the proposed PWM control method.

Example 1: Consider a resonant circuit system for liquid level control [24], which is modeled as

$$\begin{cases} \dot{\mathbf{i}}_{L_1}(t) = -\dfrac{\mathbf{v}_c(t)}{L_1} - \dfrac{R_a\left(\mathbf{i}_{L_2}(t) - 0.5\sin(\mathbf{v}_c(t))\right)}{L_1} \\ \dot{\mathbf{v}}_c(t) = \dfrac{\mathbf{i}_{L_2}(t)}{C} - \dfrac{0.5\sin(\mathbf{v}_c(t))}{C} \\ \dot{\mathbf{i}}_{L_2}(t) = -\dfrac{R_b \mathbf{i}_{L_2}(t)}{L_2} + \dfrac{\mathbf{v}(t)}{L_2} \end{cases}$$

where $\mathbf{v}(t)$ is a control input voltage, $\mathbf{v}_c(t)$ is the voltage across the capacitor C, $\mathbf{i}_{L_1}$ and $\mathbf{i}_{L_2}$ are the currents through the tunnel diode, R, R_a, and R_b are the resistances, and L_1 and L_2 are the inductors.

Let the state $x_1(t) = L_1 \mathbf{i}_{L_1}$, $x_2(t) = -\mathbf{v}_c(t)$, and $x_3 = -\frac{1}{C}(\mathbf{i}_{L_2}(t) - 0.5\sin(\mathbf{v}_c(t)))$. Choosing the prefeedback control input as $\mathbf{v}(t) = -L_2 C u - R_b C x_3 - 0.5 R_b \sin(x_2) - 0.5 L_2 \cos(x_2) x_3$, one

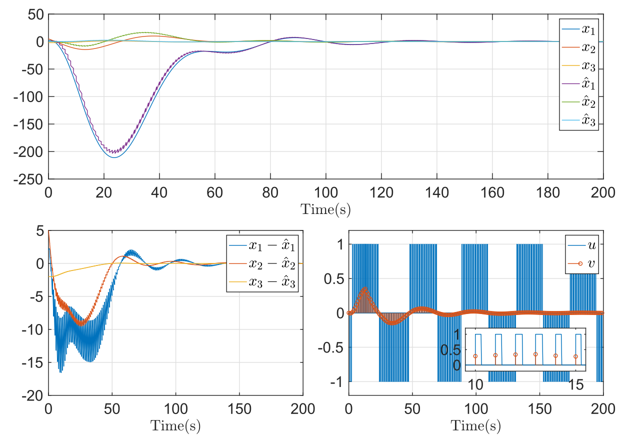

Fig. 1. Trajectory in Example 1.

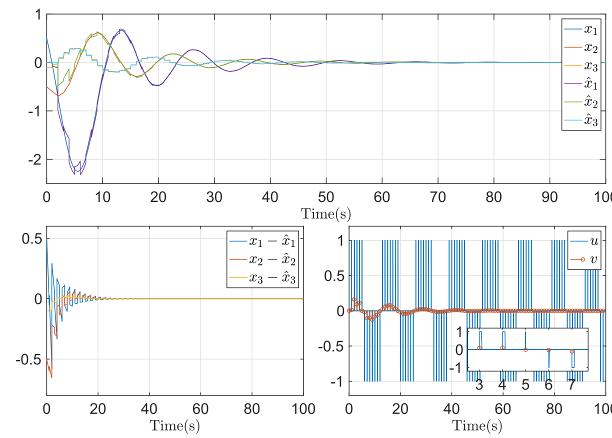

Fig. 2. Trajectory in Example 2.

has the following system model:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + R_a C x_3(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = u(t). \end{cases} \tag{28}$$

The parameters in this circuit are $L_1 = L_2 = 1\text{H}$, $C = 2\text{F}$, and $R_a = R_b = 1\Omega$. The sampled-data period T is chosen as $1\ s$, and the measurement output is given as

$$y(t_k) = x_1(t_k).$$

It can be verified that system (28) satisfies Assumption 1 with $\theta = 2$. The amplitude of the PWM signal is assumed to be $m = 1$.

Following the main result, one can choose $L = (-3, -3, -1)^T$, $K = (-0.5, -0.8, -0.6)$, and $\gamma = 2$. According to Theorem 1, the desired parameter h_0 is chosen as 6. The simulation results are shown in Fig. 1 when the initial states are $x(0) = (2, 5, -1)^T$ and $\hat{x}(0) = (0, 0, 0)^T$. It is shown that the designed observer can well estimate the circuit system's states, and both the system state and the observer state converge to 0. One can further find that the observer state is not continuous, and updated in each sampled-data instant. Furthermore, the dynamics of control inputs $u(t)$ have three states -1, 0, and 1, and switches between these three states.

Example 2: Consider the following numerical example:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - \mu(t)u^2(t)x_3(t) \\ \dot{x}_2(t) = x_3(t) + \mu(t)u^2(t) \\ \dot{x}_3(t) = u(t) \end{cases}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is the system state, $u(t)$ is the PWM control input, and the function $\mu(t)$ is time-varying and bounded by 1. The measurement output is given as

$$y(t_k) = x_1(t_k).$$

We assume the amplitude of the PWM signal as $m = 1$. Then, it is not difficult to verify that Assumption 1 is satisfied for $\theta = 1$.

One can choose $L = (-3, -3, -1)^T$ and $K = (-0.3, -0.9, -0.6)$. We now check the effectiveness of Theorem 2 under $h = 2$. By choosing the initial states as $x(0) = (0.5, -0.5, -0.1)^T$ and $\hat{x}(0) = (0, 0, 0)^T$, we show the simulation results in Fig. 2. It is clear that the convergence of both the system state and the observer state is guaranteed. The control input signal $u(t)$ is given and it is in a rectangular pulse wave, which reflects the feature of the designed PWM controller.

VI. Conclusion

This article presented a novel PWM control design scheme for feedforward nonlinear systems, which involved a state observer and a reference signal with a regulation parameter. Particularly, in light of the constructed hybrid observer, an effective PWM control signal was generated based on the constructed reference signal. Furthermore, an equivalent discretization version of the closed-loop system was developed to handle the challenges induced by signal switching, where the influence of nonlinear residuals was restrained via the selected parameter. By virtue of the Lyapunov stability theory, sufficient conditions were deduced to preserve the stability of the closed-loop system. Finally, two examples were presented to demonstrate the efficacy of the proposed method.

Appendix

A. Proof of Proposition 1

Denote $\eta = H_k\hat{x}$ and $\epsilon = H_k e$. It follows from (5a) and (12) that:

$$\begin{cases} \dot{\eta}(t) = \dfrac{1}{h_k}A\eta(t) + \dfrac{1}{h_k}Bms_k, \\ \dot{\epsilon}(t) = \dfrac{1}{h_k}A\epsilon(t) + H_k F_{xu}, \end{cases} \quad t \in [t_k, t_k + \delta_k)$$

and

$$\begin{cases} \dot{\eta}(t) = \dfrac{1}{h_k}A\eta(t), \\ \dot{\epsilon}(t) = \dfrac{1}{h_k}A\epsilon(t) + H_k F_{xu}, \end{cases} \quad t \in [t_k + \delta_k, t_{k+1}).$$

In what follows, the time variable t will be omitted whenever without confusion.

From Assumption 1, it holds

$$\begin{aligned} &\frac{1}{h_k^{n+1-i}}\left|f_i(x_{i+2}, \ldots, x_n, u)\right| \\ &\le \frac{1}{h_k^2}\theta\left(|\eta_{i+2} + \epsilon_{i+2}| + \ldots + |\eta_n + \epsilon_n| + |u|\right) \\ &\le \frac{1}{h_k^2}\theta\left(\sqrt{n}\left(\|\eta\| + \|\epsilon\|\right) + |u|\right) \end{aligned}$$

for $i = 1, 2, \ldots, n-1$, $\frac{1}{h_k^2}|f_{n-1}(u)| \le \frac{1}{h_k^2}\theta|u|$, and $f_n(u) = 0$, which can be utilized to further deduce

$$\|H_k F\| \le \frac{1}{h_k^2}\theta(n(\|\eta\| + \|\epsilon\|) + \sqrt{n}\,|u|).$$

Because of $|u| \le m$, we get that, for $t \in [t_k, t_k + \delta_k)$

$$\begin{cases} \dfrac{\mathrm{d}\,\|\eta\|}{\mathrm{d}t} \le \|A\|\,\|\eta\| + m\,\|B\| \\ \dfrac{\mathrm{d}\,\|\epsilon\|}{\mathrm{d}t} \le (\|A\| + \theta n)\,\|\epsilon\| + \theta n\,\|\eta\| + m\theta\sqrt{n}. \end{cases}$$

It can be further computed as

$$\begin{aligned} \|\eta(t)\| &\le e^{\|A\|(t-t_k)}\|\eta(t_k)\| + m\int_{t_k}^{t} e^{\|A\|(t-s)}\|B\|ds \\ &\le e^{\|A\|\delta_k}\|\eta(t_k)\| + \frac{e^{\|A\|\delta_k} - 1}{\delta_k}\frac{\|B\|T}{\|A\|}\frac{m\delta_k}{T}. \end{aligned} \tag{29}$$

Similarly, it is not difficult to calculate that

$$\begin{aligned} \|\epsilon(t)\| \le\, & e^{(\|A\|+\theta n)\delta_k}\|\epsilon(t_k)\| + e^{(\|A\|+\theta n)\delta_k}\|\eta(t_k)\| \\ & + e^{\|A\|\delta_k}\frac{e^{\theta n\delta_k} - 1}{\delta_k}\frac{\|B\|T}{\|A\|}\frac{m\delta_k}{T} \\ & + \frac{\theta\sqrt{n}T}{\|A\| + \theta n}\frac{e^{(\|A\|+\theta n)\delta_k} - 1}{\delta_k}\frac{m\delta_k}{T}. \end{aligned} \tag{30}$$

On the other hand, one can obtain, for $t \in [t_k + \delta_k, t_{k+1})$,

$$\begin{cases} \dfrac{\mathrm{d}\,\|\eta\|}{\mathrm{d}t} \le \|A\|\,\|\eta\| \\ \dfrac{\mathrm{d}\,\|\epsilon\|}{\mathrm{d}t} \le (\|A\| + \theta n)\,\|\epsilon\| + \theta n\,\|\eta\| \end{cases}$$

and further has

$$\begin{cases} \|\eta(t)\| \le e^{\|A\|T}\|\eta(t_k)\| + e^{\|A\|T}\frac{e^{\|A\|\delta_k}-1}{\delta_k}\frac{\|B\|T}{\|A\|}\frac{m\delta_k}{T} \\ \|\epsilon(t)\| \le e^{(\|A\|+\theta n)T}\|\epsilon(t_k)\| + 2e^{(\|A\|+\theta n)T}\|\eta(t_k)\| \\ \qquad + Te^{(\|A\|+\theta n)T}\frac{\theta\sqrt{n}+2\|B\|}{\|A\|}\frac{e^{(\|A\|+\theta n)\delta_k}-1}{\delta_k}\frac{m\delta_k}{T}. \end{cases} \tag{31}$$

It is worth mentioning that, when $|v_k| \le m$, it holds

$$\begin{aligned} & m\frac{\delta_k}{T} = |v_k| \le \|K\|\,\|H_k\hat{x}\,(t_k)\| \le \|K\|\|\eta(t_k)\| \\ & \|\mathscr{H}_k Z_k\|^2 = \|\epsilon(t_k)\|^2 + \|\eta(t_k)\|^2. \end{aligned} \tag{32}$$

Combining (29), (30), (31), and (32), and considering the boundedness of $(e^{\tau s} - 1)/s$ for $s \in (0, T]$, a positive constant ϑ_0 can be found such that (17) holds. This ends the proof.

B. Proof of Proposition 2

It is noted that

$$\begin{aligned} & \|H_k\mathcal{F}_k\| \\ & \le \int_{t_k}^{t_{k+1}} \exp\left(\frac{1}{h_k}\|A\|\,(t_{k+1} - s)\right)\|H_k F\|\,ds \\ & \le \exp\left(\|A\|\,T\right)\frac{T}{h_k^2}\theta n \max_{t\in[t_k, t_{k+1})}\{\|\epsilon(t)\| + \|\eta(t)\|\} \\ & \quad + \frac{T}{h_k^2}\theta\sqrt{n}\exp\left(\|A\|\,T\right)\frac{1}{\|A\|}\frac{e^{\|A\|\delta_k} - 1}{\delta_k}\frac{m\delta_k}{T}. \end{aligned}$$

By employing Proposition 1, we can find a constant $\bar{\vartheta}$ such that $\|H_k\mathcal{F}_k\| \le \alpha_k^2\bar{\vartheta}\|\mathscr{H}_k Z_k\|$. Furthermore, due to $H_k\mathcal{F}_{1\,k} = (I + \alpha_k LC)H_k\mathcal{F}_k$ and $H_k\mathcal{F}_{2\,k} = \alpha_k LCH_k\mathcal{F}_k$, we have $\|H_k\mathcal{F}_{1\,k}\| \le (\|I\| + \|LC\|)\|H_k\mathcal{F}_k\|$ and $\|H_k\mathcal{F}_{2\,k}\| \le \|LC\|\|H_k\mathcal{F}_k\|$. Recalling $\mathscr{F}_{1\,k}$ in (15), there exists a constant ϑ such that (18) holds, which ends the proof.

C. Proof of Proposition 3

Considering $h_k \ge T \ge \delta_k$, we have $T/h_k = \alpha_k$ and $ms_k\delta_k/T = v_k$. Since

$$\int_0^{\delta_k}\exp\left(-\frac{1}{h_k}As\right)\frac{1}{h_k}Bds = \begin{pmatrix} -\frac{1}{(n+1)!}\left(-\frac{\delta_k}{h_k}\right)^n \\ \vdots \\ -\frac{1}{2!}\left(-\frac{\delta_k}{h_k}\right)^2 \\ \frac{\delta_k}{h_k} \end{pmatrix}$$

one has

$$\begin{aligned} \mathcal{F}_{3\,k} =\, & \exp\left(A\alpha_k\right) \\ & \times\left(\int_0^{\delta_k}\exp\left(-\frac{1}{h_k}As\right)\frac{1}{h_k}Bds\cdot ms_k - \alpha_k Bv_k\right) \\ =\, & \exp\left(A\alpha_k\right)\begin{pmatrix} -\frac{1}{(n+1)!}\left(-\frac{\delta_k}{h_k}\right)^{n-2}\frac{T^2}{h_k^2}\frac{\delta_k}{T}\cdot ms_k\frac{\delta_k}{T} \\ \vdots \\ -\frac{1}{2!}\frac{T^2}{h_k^2}\frac{\delta_k}{T}\cdot ms_k\frac{\delta_k}{T} \\ 0 \end{pmatrix}. \end{aligned}$$

Then, it

$$\begin{aligned} \|\mathcal{F}_{3\,k}\| & \le \alpha_k^2\frac{\delta_k}{T}\exp\left(\bar{\alpha}\,\|A\|\right)\sqrt{n-1}\frac{1}{2}\,|v_k| \\ & \le \alpha_k^2 d_k\exp\left(\bar{\alpha}\,\|A\|\right)\sqrt{n-1}\frac{1}{2}\,\|K\|\,\|H_k\hat{x}(t_k)\|\,. \end{aligned}$$

From $\mathscr{F}_{2\,k}$ in (15), we get (18), which ends the proof.

D. Proof of Lemma 1

The proof is similar to that in [31] and [32], and hence its basic idea is presented here. Introduce a function $\omega(v, \alpha) = v^T\mathscr{L}^T(\alpha)P\mathscr{L}(\alpha)v$ with $v^Tv = 1$, and then calculate its derivative as follows:

$$\left.\frac{\partial\omega}{\partial\alpha}\right|_{\alpha=0} = v^T\mathscr{A}^TPv + v^TP\mathscr{A}v \le -\varrho.$$

Then, there is a constant $\bar{\alpha}$ such that $\alpha \le \bar{\alpha}$ and $\frac{\partial\omega}{\partial\alpha} \le -\frac{\varrho}{2}$. When $\alpha \in (0, \bar{\alpha}]$, it holds

$$\omega\left(v, \alpha\right) \le \omega(v, 0) - \alpha\frac{\varrho}{2}.$$

Because of $\omega(v, 0) = v^TPv$, one gets

$$v^T\mathscr{L}^T\left(\alpha\right)P\mathscr{L}\left(\alpha\right)v - v^TPv \le -\alpha\frac{\varrho}{2}$$

for any $v \in \{v|v^Tv = 1\}$, which implies

$$\mathscr{L}^T\left(\alpha\right)P\mathscr{L}\left(\alpha\right) - P \le -\alpha\frac{\varrho}{2}.$$

Furthermore, one gets

$$\begin{aligned}\|\mathscr{L}(\alpha)\|^2 &= \|\Phi_1(\alpha)\|^2 + \alpha^2\|LC\exp(A\alpha)\|^2 + \|\Phi_2(\alpha)\|^2 \\ &\le (\|I\| + \bar{\alpha}\|LC\|)^2 \exp(2\bar{\alpha}\|A\|) \\ &\quad + \exp(2\bar{\alpha}\|A\|)(\|I\| + \bar{\alpha}\|BK\|)^2 \\ &\quad + \bar{\alpha}^2\|LC\|^2 \exp(2\bar{\alpha}\|A\|).\end{aligned}$$

One can find a constant ρ to meet (19), which ends the proof.

E. Proof of Lemma 2

Consider $\omega(v,\hbar) = \hbar^{-\sigma} v^T \bar{\mathcal{H}} P \bar{\mathcal{H}} v$ with $v^T v = 1$. Then, it holds

$$\begin{aligned}\frac{\partial\omega(v,\hbar)}{\partial\hbar} &= \frac{1}{\hbar^{1+\sigma}} v^T \bar{\mathcal{H}} (P\mathscr{D} + \mathscr{D}P) \bar{\mathcal{H}} v \\ &\quad - \sigma \frac{1}{\hbar^{1+\sigma}} v^T \bar{\mathcal{H}} P \bar{\mathcal{H}} v \ge 0\end{aligned}$$

where (8) is employed. Thus, we obtain

$$\omega(v,1) \ge \omega(v,\hbar), \quad \text{for all} \quad \hbar \le 1.$$

Since $\omega(v,1) = v^T P v$, we get (20). This ends the proof.

References

[1] A. Komaee, "Stabilization of linear systems by pulsewidth modulation of switching actuators," *IEEE Trans. Autom. Control*, vol. 65, no. 5, pp. 1969–1984, May 2020.

[2] L. Hou and A. N. Michel, "Stability analysis of pulse-width-modulated feedback systems," *Automatica*, vol. 37, no. 9, pp. 1335–1349, Sep. 2001.

[3] W. An, P. Zhao, H. Liu, and J. Hu, "Distributed multi-step subgradient projection algorithm with adaptive event-triggering protocols: A framework of multiagent systems," *Int. J. Syst. Sci.*, vol. 53, no. 13, pp. 2758–2772, 2022.

[4] T. Kadota and H. Bourne, "Stability conditions of pulse-width-modulated systems through the second method of Lyapunov," *IRE Trans. Autom. Control*, vol. 6, no. 3, pp. 266–276, Sep. 1961.

[5] T. Liu and Z.-P. Jiang, "Distributed control of multi-agent systems with pulse-width-modulated controllers," *Automatica*, vol. 119, Sep. 2020, Art. no. 109020.

[6] S.-L. Jung and Y.-Y. Tzou, "Discrete sliding-mode control of a PWM inverter for sinusoidal output waveform synthesis with optimal sliding curve," *IEEE Trans. Power Electron.*, vol. 11, no. 4, pp. 567–577, Jul. 1996.

[7] E.-C. Chang, Y.-C. Liu, and C.-H. Chang, "Experimental performance comparison of various sliding modes controlled PWM inverters," *Energy Procedia*, vol. 156, pp. 110–114, Jan. 2019.

[8] A. G. Beccuti et al., "Hybrid control techniques for switched-mode DC-DC converters part II: The step-up topology," in *Proc. Amer. Control Conf.*, 2007, pp. 5464–5471.

[9] S. Mariethoz et al., "Comparison of hybrid control techniques for buck and boost DC–DC converters," *IEEE Trans. Control Syst. Technol.*, vol. 18, no. 5, pp. 1126–1145, Sep. 2010.

[10] H. Fujioka et al., "LQ optimal control for a class of pulse width modulated systems," *Automatica*, vol. 43, no. 6, pp. 1009–1020, Jun. 2007.

[11] W. L. D. Koning, "Digital optimal reduced-order control of pulse-width-modulated switched linear systems," *Automatica*, vol. 39, no. 11, pp. 1997–2003, Nov. 2003.

[12] L. Praly, R. Ortega, and G. Kaliora, "Stabilization of nonlinear systems via forwarding $\mod \{l_g V\}$," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1461–1466, Sep. 2001.

[13] A. Zavala-Río, I. Fantoni, and R. Lozano, "Global stabilization of a PVTOL aircraft model with bounded inputs," *Int. J. Control*, vol. 76, no. 18, pp. 1833–1844, Dec. 2003.

[14] H. Ye, "Saturated delayed controls for feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1646–1653, Jun. 2014.

[15] A. Teel, "Using saturation to stabilize a class of single-input partially linear composite systems," *IFAC Proc. Vol.*, vol. 25, no. 13, pp. 379–384, Jun. 1992.

[16] H. Li, X. Zhang, and S. Liu, "An improved dynamic gain method to global regulation of feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 67, no. 6, pp. 2981–2988, Jun. 2022.

[17] F. Yang, J. Li, H. Dong, and Y. Shen, "Proportional-integral-type estimator design for delayed recurrent neural networks under encoding-decoding mechanism," *Int. J. Syst. Sci.*, vol. 53, no. 13, pp. 2729–2741, 2022.

[18] X. Zhang et al., "Feedback stabilization for high order feedforward nonlinear time-delay systems," *Automatica*, vol. 47, no. 5, pp. 962–967, May 2011.

[19] B. Zhou and X. Yang, "Global stabilization of feedforward nonlinear time-delay systems by bounded controls," *Automatica*, vol. 88, pp. 21–30, Feb. 2018.

[20] C. Zhao and W. Lin, "Memoryless linear feedback control for a class of upper-triangular systems with large delays in the state and input," *Syst. Control Lett.*, vol. 139, 2020, Art. no. 104679.

[21] F. Mazenc, S. Mondié, and S.-I. Niculescu, "Global asymptotic stabilization for chains of integrators with a delay in the input," *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 57–63, Jan. 2003.

[22] F. Mazenc, S. Mondie, and R. Francisco, "Global asymptotic stabilization of feedforward systems with delay in the input," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 844–850, May 2004.

[23] J. Sun and W. Lin, "A dynamic gain-based saturation control strategy for feedforward systems with long delays in state and input," *IEEE Trans. Autom. Control*, vol. 66, no. 9, pp. 4357–4364, Sep. 2021.

[24] M.-S. Koo, H.-L. Choi, and J.-T. Lim, "Global regulation of a class of uncertain nonlinear systems by switching adaptive controller," *IEEE Trans. Autom. Control*, vol. 55, no. 12, pp. 2822–2827, Dec. 2010.

[25] X. Ye and H. Unbehauen, "Global adaptive stabilization for a class of feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 786–792, May 2004.

[26] Q. Liu, X. Zhang, and H. Li, "Global regulation for feedforward systems with both discrete delays and distributed delays," *Automatica*, vol. 113, 2020, Art. no. 108753.

[27] X. Zhang, L. Baron, Q. Liu, and E.-K. Boukas, "Design of stabilizing controllers with a dynamic gain for feedforward nonlinear time-delay systems," *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 692–697, Mar. 2011.

[28] X. Jia, S. Xu, and S. Zhou, "Adaptive output feedback control of nonlinear systems: A dynamic-gain scaling approach," *IEEE Trans. Autom. Control*, vol. 68, no. 8, pp. 5150–5157, Aug. 2023.

[29] H. Du, C. Qian, Y. He, and Y. Cheng, "Global sampled-data output feedback stabilisation of a class of upper-triangular systems with input delay," *IET Control Theory Appl.*, vol. 7, no. 10, pp. 1437–1446, Jul. 2013.

[30] P. Wang, C. Yu, and J. Sun, "Quantized feedback control for nonlinear feedforward systems with unknown output functions and unknown control coefficients," *Int. J. Robust Nonlinear Control*, vol. 29, no. 12, pp. 4002–4021, Aug. 2019.

[31] J. Peralez, V. Andrieu, M. Nadri, and U. Serres, "Event-triggered output feedback stabilization via dynamic high-gain scaling," *IEEE Trans. Autom. Control*, vol. 63, no. 8, pp. 2537–2549, Aug. 2018.

[32] V. Andrieu et al., "Self-triggered continuous discrete observer with updated sampling period," *Automatica*, vol. 62, pp. 106–113, Dec. 2015.